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An energy principle for dissipative fluids

By E. M. BARSTON
Courant Institute of Mathematical Sciences, New York University

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An energy principle is presented which gives necessary and sufficient conditions for exponential stability for a large class of dissipative systems. The maximal growth rate Ω of an unstable system is shown to be the least upper bound of a certain functional, giving a variational expression for Ω. These results are used to discuss the gravitational stability of incompressible viscous fluids and resistive magnetofluids in arbitrary geometries.

1. Introduction

The utility of the energy principle in determining the stability of equilibria of conservative dynamical systems is well known (Chandrasekhar 1961). In this paper we extend the energy principle to a large class of dissipative systems, and in the process obtain a 'maximum' principle for the maximal growth rate of an unstable system. The latter permits the use of a variational or Rayleigh–Ritz technique to calculate the maximal growth rate.

The paper begins with a discussion of two stability problems which are used to motivate as well as illustrate the theory, viz. the gravitational stability of a viscous incompressible fluid (§2) and the gravitational stability of a viscous, resistive, incompressible magnetofluid (§3). The equilibria considered are quite general. The necessary notation and equations are developed and the equations put into canonical form (4.1) in §§2 and 3. The energy principle and the maximum principle are derived in §4, and are then applied to the original problems in §5. While the necessary and sufficient conditions for exponential stability for the first of these problems is physically apparent ($\nabla\rho_0 . \nabla\Phi_0 \leqslant 0$ where ρ_0 and Φ_0 are, respectively, the unperturbed mass density and gravitational potential), the situation is somewhat more complex for the conducting fluid, and depends also upon conductivity, magnetic field, and geometry. Variational expressions for the maximal growth rates are given for both problems and permit the study of this quantity as a function of viscosity, geometry, conductivity, and magnetic field. Details will be found in §5.

It must be emphasized that the extended energy principle and the maximum principle apply to any system that can be put into the canonical form of (4.1), and are by no means limited in application to the specific examples treated in this paper. Once the proper form is achieved, necessary and sufficient conditions for exponential stability follow immediately, and the calculation of the maximal growth rate (a function of the parameters of the unstable system) becomes a straightforward variational problem. Thus the successful manipulation of a problem into the form of (4.1) is well rewarded. However, the proper choice of variables is not always obvious. The theory has proved extremely successful in treating the resistive magnetohydrodynamic sheet pinch and the electrohydrodynamic Rayleigh–Taylor bulk instability, providing growth rates as well as necessary and sufficient conditions for stability where only partial results for special limiting cases had previously existed, some of which were shown to be incorrect (Barston 1969*a*, 1970; Turnbull & Melcher 1969).

The derivation of the extended energy principle used herein has the advantage of being free from any assumptions of completeness imposed on the eigenfunctions of the linearized perturbation equations; in fact, our results are valid for systems with no proper eigenfunctions. This is important in certain applications to hydrodynamic and hydromagnetic systems, where the set of proper eigenfunctions is not complete. We make the much weaker assumption that the time-dependent perturbation equations admit smooth solutions for all sufficiently smooth initial data (we do not require the existence of *any* solutions of the form $f(\mathbf{x})\, e^{\omega t}$), which is a rather obvious requirement to be made on any well-posed physical system.

2. Equations for a viscous fluid

We consider a viscous incompressible fluid occupying a volume U with surface S, satisfying the following system of equations in U:

$$\nabla . \mathbf{v} = 0, \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} + \mathbf{v} . \nabla \rho = 0, \tag{2.2}$$

$$\rho \left\{ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} . \nabla)\mathbf{v} \right\} = -\nabla P - \rho \nabla \Phi_0 + \nabla^2(\nu \mathbf{v}) - \mathbf{v}\nabla^2 \nu - (\nabla \times \mathbf{v}) \times \nabla \nu. \tag{2.3}$$

The quantity $\rho(\mathbf{x}, t)$ denotes the mass density, $\mathbf{v}(\mathbf{x}, t)$ the fluid velocity, $P(\mathbf{x}, t)$ the scalar pressure, $\nu(\mathbf{x}, t)$ the viscosity, and $\Phi_0(\mathbf{x})$ the (external) gravitational potential. (We neglect the self-gravitational field.)

The equilibrium values of the fluid variables, denoted by a 0 subscript, are as follows: $\mathbf{v}_0(\mathbf{x}) \equiv 0$, $\nu_0(\mathbf{x})$ is an arbitrary prescribed positive function, while $P_0(\mathbf{x})$, $\rho_0(\mathbf{x})$, and $\Phi_0(\mathbf{x})$ are related by

$$\nabla P_0 = -\rho_0 \nabla \Phi_0, \tag{2.4}$$

but ρ_0 and Φ_0 are otherwise arbitrary (ρ_0 will be assumed strictly positive).

We linearize (2.1)–(2.3) about the equilibrium and obtain, after introducing the displacement vector

$$\boldsymbol{\xi}(\mathbf{x}, t) \equiv \int_0^t \mathbf{v}_1(\mathbf{x}, \tau)\, d\tau + \boldsymbol{\xi}(\mathbf{x}, 0), \tag{2.5}$$

the following set of equations (the linearized variables are denoted by a subscript 1; note that $\boldsymbol{\xi}$ represents the linearized displacement):

$$\rho_1 = \rho_1(\mathbf{x}, 0) - \nabla \rho_0 . [\boldsymbol{\xi} - \boldsymbol{\xi}(\mathbf{x}, 0)], \tag{2.6}$$

$$\mathbf{v}_1 = \frac{\partial \boldsymbol{\xi}}{\partial t}, \tag{2.7}$$

$$\nabla . \boldsymbol{\xi} = 0, \tag{2.8}$$

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} + L_0 \frac{\partial \boldsymbol{\xi}}{\partial t} + G_0 \boldsymbol{\xi} + \boldsymbol{\nabla} P_1 = -[\rho_1(\mathbf{x}, 0) + \boldsymbol{\nabla}\rho_0 . \boldsymbol{\xi}(\mathbf{x}, 0)] \boldsymbol{\nabla}\Phi_0, \tag{2.9}$$

where
$$L_0 \frac{\partial \boldsymbol{\xi}}{\partial t} \equiv -\nabla^2\left(\nu_0 \frac{\partial \boldsymbol{\xi}}{\partial t}\right) - \boldsymbol{\nabla}\nu_0 \times \left(\nabla \times \frac{\partial \boldsymbol{\xi}}{\partial t}\right) + (\nabla^2 \nu_0) \frac{\partial \boldsymbol{\xi}}{\partial t} \tag{2.10}$$

and
$$G_0 \boldsymbol{\xi} \equiv -\boldsymbol{\nabla}\Phi_0(\boldsymbol{\nabla}\rho_0 . \boldsymbol{\xi}). \tag{2.11}$$

The appropriate boundary condition is that $\boldsymbol{\xi}(\mathbf{x}, t)$ vanish on S. We assume, of course, that all quantities are sufficiently smooth so that the indicated operations are well defined; in particular, we take Φ_0, ρ_0 and ν_0 to be twice continuously differentiable functions and consider the class C of solutions $\boldsymbol{\xi}(\mathbf{x}, t)$ of (2.9) such that $\boldsymbol{\xi}(\mathbf{x}, t)$ and $\{\partial \boldsymbol{\xi}(\mathbf{x}, t)\}/\partial t$ are both in D for each fixed $t \geqslant 0$, where D is defined to be the set of all functions $\mathbf{f}(\mathbf{x})$ with the properties that $\nabla . \mathbf{f} = 0$ in U, $\mathbf{f} = 0$ on S, and $\mathbf{f}$ is twice continuously differentiable on U.

It is not difficult to show that the operators L_0 and G_0 defined by (2.10) and (2.11) are self-adjoint on D with respect to the inner product,

$$(\mathbf{f}, \mathbf{g}) \equiv \int_U \mathbf{f}^* . \mathbf{g} \, d^3x \tag{2.12}$$

($\mathbf{f}^*$ denotes the complex conjugate of $\mathbf{f}$), and we have, using standard Cartesian tensor notation,

$$(\boldsymbol{\xi}, L_0 \boldsymbol{\xi}) = \frac{1}{2} \int_U \nu_0 \left| \frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} \right|^2 d^3x,$$

so that L_0 is positive. From (2.4),

$$0 = \nabla \times (\rho_0 \boldsymbol{\nabla}\Phi_0) = \boldsymbol{\nabla}\rho_0 \times \boldsymbol{\nabla}\Phi_0,$$

which implies the existence of a scalar function $\gamma_0(\mathbf{x})$ such that

$$\boldsymbol{\nabla}\rho_0 = \gamma_0(\mathbf{x}) \boldsymbol{\nabla}\Phi_0, \quad \boldsymbol{\nabla}\Phi_0 \neq 0. \tag{2.13}$$

Therefore
$$(\boldsymbol{\xi}, G_0 \boldsymbol{\xi}) = -\int_U \boldsymbol{\nabla}\rho_0 . \boldsymbol{\nabla}\Phi_0 |\xi_\parallel|^2 \, d^3x, \tag{2.14}$$

where $\xi_\parallel \equiv (\boldsymbol{\xi} . \boldsymbol{\nabla}\Phi_0)/|\boldsymbol{\nabla}\Phi_0|$.

We assume that $\boldsymbol{\nabla}\rho_0 . \boldsymbol{\nabla}\Phi_0$ is bounded on U. For future reference, we note that

$$(\boldsymbol{\xi}, \boldsymbol{\nabla} P_1) = 0 = \left(\frac{\partial \boldsymbol{\xi}}{\partial t}, \boldsymbol{\nabla} P_1\right) \tag{2.15}$$

holds for each solution $\boldsymbol{\xi}$ in C, since

$$(\boldsymbol{\xi}, \boldsymbol{\nabla} P_1) = \int_U \boldsymbol{\xi}^* . \boldsymbol{\nabla} P_1 \, d^3x = \int_U \nabla . (P_1 \boldsymbol{\xi}^*) \, d^3x = \int_S P_1 \boldsymbol{\xi}^* . d\mathbf{S} = 0. \tag{2.16}$$

3. Equations for a viscous resistive MHD fluid

Let the fluid occupy the volume U with surface S, and satisfy the following set of equations in U:

$$\nabla . \mathbf{v} = 0, \tag{3.1}$$

$$\frac{\partial \rho}{\partial t} + \mathbf{v} . \nabla \rho = 0, \tag{3.2}$$

$$\rho\left\{\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}.\nabla)\mathbf{v}\right\} = -\boldsymbol{\nabla}P-\rho\boldsymbol{\nabla}\Phi_0+\mu_0^{-1}(\nabla\times\mathbf{B})\times\mathbf{B}+\nabla^2(\nu\mathbf{v})-\mathbf{v}\nabla^2\nu-(\nabla\times\mathbf{v})\times\boldsymbol{\nabla}\nu, \tag{3.3}$$

$$\partial\mathbf{B}/\partial t = -\mu_0^{-1}\nabla\times(\eta\nabla\times\mathbf{B})+\nabla\times(\mathbf{v}\times\mathbf{B}), \tag{3.4}$$

$$\nabla.\mathbf{B}(\mathbf{x},0) = 0. \tag{3.5}$$

The equations are written in MKS units; μ_0 is the permeability of free space. The quantity $\mathbf{B}(\mathbf{x},t)$ denotes the magnetic field, $\eta(\mathbf{x},t)$ the resistivity, and the remaining variables are as defined in § 2.

For the equilibrium, we take $\mathbf{v}_0(\mathbf{x}) = 0$, $\nabla\times\mathbf{B}_0(\mathbf{x}) = 0$ within U, while $\rho_0(\mathbf{x})$, $P_0(\mathbf{x})$ and $\Phi_0(\mathbf{x})$ must satisfy (2.4). We assume that $\rho_0(\mathbf{x})$, $\nu_0(\mathbf{x})$, and $\eta_0(\mathbf{x})$ are positive in U, that η_0 vanishes identically on all parts of S not at infinity, and that ρ_0 and ν_0 are positive on S.

The linearization of (3.1)–(3.4) gives

$$\nabla.\mathbf{v}_1 = 0, \tag{3.6}$$

$$\frac{\partial\rho_1}{\partial t}+\mathbf{v}_1.\boldsymbol{\nabla}\rho_0 = 0, \tag{3.7}$$

$$\mu_0\rho_0\frac{\partial\mathbf{v}_1}{\partial t} = -\mu_0\boldsymbol{\nabla}P_1-\mu_0\rho_1\boldsymbol{\nabla}\Phi_0-\mu_0L_0\mathbf{v}_1-L_1\mathbf{B}_1, \tag{3.8}$$

$$\frac{\partial\mathbf{B}_1}{\partial t} = -L_2\mathbf{B}_1+L_1^+\mathbf{v}_1, \tag{3.9}$$

where

$$L_1\mathbf{B} \equiv \mathbf{B}_0\times[\nabla\times\mathbf{B}], \tag{3.10}$$

$$L_2\mathbf{B} \equiv \mu_0^{-1}\nabla\times[\eta_0\nabla\times\mathbf{B}], \tag{3.11}$$

$$L_1^+\mathbf{v} \equiv -\nabla\times[\mathbf{B}_0\times\mathbf{v}], \tag{3.12}$$

and L_0 is given by (2.10). Setting

$$\mathbf{R}(\mathbf{x},t) \equiv \int_0^t \mathbf{B}_1(\mathbf{x},\tau)\,d\tau+\mathbf{R}(\mathbf{x},0), \tag{3.13}$$

and introducing the displacement vector $\boldsymbol{\xi}$ of (2.5), (3.7)–(3.9) yield (2.6) and

$$\Pi\frac{\partial^2\bar{\xi}}{\partial t^2}+K\frac{\partial\bar{\xi}}{\partial t}+H\bar{\xi}+F_{\bar{\xi}} = \bar{r}, \tag{3.14}$$

where $\bar{\xi}(\mathbf{x},t)$ and $\bar{r}(\mathbf{x})$ denote the six-vectors

$$\bar{\xi}(\mathbf{x},t) = \begin{pmatrix}\boldsymbol{\xi}_1(\mathbf{x},t)\\ \boldsymbol{\xi}_2(\mathbf{x},t)\end{pmatrix} = \begin{pmatrix}\boldsymbol{\xi}(\mathbf{x},t)\\ \mathbf{R}(\mathbf{x},t)\end{pmatrix}, \tag{3.15}$$

$$\bar{r}(\mathbf{x}) = \begin{pmatrix}-\mu_0\boldsymbol{\nabla}\Phi_0[\rho_1(\mathbf{x},0)+\boldsymbol{\nabla}\rho_0.\boldsymbol{\xi}(\mathbf{x},0)]-L_1[\mathbf{B}_1(\mathbf{x},0)+L_2\mathbf{R}(\mathbf{x},0)-L_1^+\boldsymbol{\xi}(\mathbf{x},0)]\\ L_2[\mathbf{B}_1(\mathbf{x},0)+L_2\mathbf{R}(\mathbf{x},0)-L_1^+\boldsymbol{\xi}(\mathbf{x},0)]\end{pmatrix}, \tag{3.16}$$

$$\Pi = \begin{pmatrix}\mu_0\rho_0 & 0\\ 0 & 0\end{pmatrix},\quad K = \begin{pmatrix}\mu_0L_0 & 0\\ 0 & L_2\end{pmatrix},\quad H = \begin{pmatrix}L_1L_1^++\mu_0G_0 & -L_1L_2\\ -L_2L_1^+ & L_2^2\end{pmatrix}, \tag{3.17}$$

and

$$F_{\bar{\xi}} = \begin{pmatrix}\mu_0\boldsymbol{\nabla}P_1\\ 0\end{pmatrix}. \tag{3.18}$$

We take S to be a perfectly conducting rigid surface, and require that $\boldsymbol{\xi}_1(\mathbf{x},t)$ $(= \boldsymbol{\xi}(\mathbf{x},t))$ vanish on S, $\boldsymbol{\xi}_2(\mathbf{x},t)$ $(= \mathbf{R}(\mathbf{x},t))$ vanish on all parts of S at infinity, while $\boldsymbol{\xi}_2 . \mathbf{n} = 0$ on the remainder of S ($\mathbf{n}$ denotes the unit normal on S). We assume that all quantities are sufficiently smooth so that the indicated operations are well defined; in particular, we take all 0-order quantities to be three times continuously differentiable, and consider the class C of solutions of (3.14) such that $\bar{\xi}(\mathbf{x},t)$ and $\partial\bar{\xi}(\mathbf{x},t)/\partial t$ are both in D for each fixed $t \geqslant 0$, where D is defined to be the set of all six-vectors

$$\bar{f}(\mathbf{x}) = \begin{pmatrix}\mathbf{f}_1(\mathbf{x})\\ \mathbf{f}_2(\mathbf{x})\end{pmatrix}$$

with the following properties: $\nabla . \mathbf{f}_1(\mathbf{x}) = 0$ in U $(i = 1,2)$, $\mathbf{f}_1$ vanishes on S, $\mathbf{f}_2$ vanishes on all parts of S at infinity and $\mathbf{f}_2 . \mathbf{n} = 0$ on the remainder of S, $L_1^{+}\mathbf{f}_1 - L_2\mathbf{f}_2$ vanishes on all parts of S at infinity and $(L_1^{+}\mathbf{f}_1 - L_2\mathbf{f}_2) . \mathbf{n} = 0$ on the remainder of S, $\mathbf{f}_1$ is three times continuously differentiable on U, and $\mathbf{f}_2$ is four times continuously differentiable on U. Under these circumstances it is easy to show that (3.14) is equivalent to (3.6)–(3.9), that Π, K, and H are self-adjoint on D with respect to the inner product $(\bar{\xi}, \bar{\zeta})$ defined by

$$(\bar{\xi}, \bar{\zeta}) \equiv \int_U \boldsymbol{\xi}_1^* . \boldsymbol{\zeta}_1 d^3x + \int_U \boldsymbol{\xi}_2^* . \boldsymbol{\zeta}_2 d^3x, \tag{3.19}$$

and we have

$$(\bar{\xi}, K\bar{\xi}) = \mu_0 \int_U \boldsymbol{\xi}_1^* . L_0 \boldsymbol{\xi}_1 d^3x + \mu_0^{-1} \int_U \eta_0 |\nabla \times \boldsymbol{\xi}_2|^2 d^3x \geqslant 0, \tag{3.20}$$

$$(\bar{\xi}, H\bar{\xi}) = \mu_0 \int_U \boldsymbol{\xi}_1^* . G_0 \boldsymbol{\xi}_1 d^3x + \int_U |L_1^{+} \boldsymbol{\xi}_1 - L_2 \boldsymbol{\xi}_2|^2 d^3x. \tag{3.21}$$

Furthermore, for every solution $\boldsymbol{\xi}(\mathbf{x},t)$ of (3.14) in C,

$$(\bar{\xi}, F_{\bar{\xi}}) = 0 = \left(\frac{\partial \bar{\xi}}{\partial t}, F_{\bar{\xi}}\right). \tag{3.22}$$

4. Stability theorems

The preceding problems have been reduced to special cases of the equation

$$P\ddot{\xi} + K\dot{\xi} + H\xi(t) + F_\xi = r(t) \quad (t \geqslant 0) \tag{4.1}$$

$$\dot{\xi} \equiv \frac{\partial \xi}{\partial t}, \quad \ddot{\xi} \equiv \frac{\partial^2 \xi}{\partial t^2},$$

where r, ξ, $\dot{\xi}$, $\ddot{\xi}$, and F_ξ are elements of an inner product space E for each fixed $t \geqslant 0$, and r is a given function of t; $\xi(t)$ and $\dot{\xi}(t)$ are, for each fixed $t \geqslant 0$, elements of a subspace D of E on which the operators K and H are self adjoint, $K \geqslant 0$ and H is bounded below; P is a non-negative self-adjoint operator with domain containing D; and F_ξ, defined only for solutions ξ of (4.1), has the property that $(\xi, F_\xi) = 0 = (\dot{\xi}, F_\xi)$ for every such ξ.

We now prove several theorems concerning the stability of solutions of (4.1), assuming the properties stated in the preceding paragraph. These theorems are straightforward modifications of results previously obtained by the author for (4.1) with F_ξ absent (Barston 1969*a*, *b*), and give necessary and sufficient conditions for exponential stability as well as maximal growth rates for the problems considered in §§2 and 3.

Stability of the solutions $\xi(t)$ of (4.1) will be discussed in terms of $\|\xi\| = (\xi, \xi)^{\frac{1}{2}}$. We say that a real or complex-valued function $f(t)$ defined for $t \geqslant 0$ is exponentially stable if for every $\epsilon > 0$, there exists a constant M_ϵ such that $|f(t)| \leqslant M_\epsilon e^{\epsilon t}$ for $t \geqslant 0$. A function $\zeta(t)$ with values in E is said to be exponentially stable if $\|\zeta(t)\|$ is exponentially stable.

The following theorem gives sufficient conditions for stability:

Theorem 4.1. Let H be non-negative and $\xi(t)$ be any solution of (4.1).

(i) If $\|r(t)\| \leqslant M$, $\|\dot{r}(t)\| \leqslant N$ for $t \geqslant 0$, where M and N are constants, and if

$$\delta \equiv \inf_D \frac{(\zeta, H\zeta)}{(\zeta, \zeta)} > 0,$$

then there exist positive constants A, B, C, Δ, and G such that

$$\|\xi(t)\| \leqslant At + B \quad (t \geqslant 0), \tag{4.2}$$

$$(\dot{\xi}, P\dot{\xi}) \leqslant Ct^2 + \Delta t + G \quad (t \geqslant 0). \tag{4.3}$$

In particular, if $N = 0$ (i.e. $\dot{r} \equiv 0$), then $\|\xi(t)\|$ and $(\dot{\xi}, P\dot{\xi})$ are bounded for $t \geqslant 0$.

(ii) If $r(t)$ and $\dot{r}(t)$ are exponentially stable and if

$$\inf_D \frac{(\zeta, [\alpha^2 P + \alpha K + H]\zeta)}{(\zeta, \zeta)} > 0 \quad \text{for all} \quad \alpha > 0,$$

then ξ and $(\dot{\xi}, P\dot{\xi})$ are exponentially stable.

(iii) If $r \equiv 0$, then $(\dot{\xi}, P\dot{\xi})$ is bounded for $t \geqslant 0$.

Proof

(i)
$$\begin{aligned}\frac{d}{dt}\{(\dot{\xi}, P\dot{\xi}) + (\xi, H\xi)\} &= (P\ddot{\xi} + H\xi, \dot{\xi}) + (\dot{\xi}, P\ddot{\xi} + H\xi) \\ &= (r - F_\xi - K\dot{\xi}, \dot{\xi}) + (\dot{\xi}, r - F_\xi - K\dot{\xi}) = 2\,\mathrm{Re}\,(r, \dot{\xi}) - 2(\dot{\xi}, K\dot{\xi}) \\ &\leqslant 2\,\mathrm{Re}\,(r, \dot{\xi}) = \frac{d}{dt}2\,\mathrm{Re}\,(r, \xi) - 2\,\mathrm{Re}\,(\dot{r}, \xi).\end{aligned} \tag{4.4}$$

Integrating from 0 to t and using the Schwarz inequality, we obtain

$$(\dot{\xi}, P\dot{\xi}) + (\xi, H\xi) \leqslant \beta + 2\|r\|\,\|\xi\| + 2\int_0^t \|\dot{r}\|\,\|\xi\|\,du, \tag{4.5}$$

where $\beta \equiv (\dot{\xi}_0, P\dot{\xi}_0) + (\xi_0, H\xi_0) - 2\,\mathrm{Re}\,(r_0, \xi_0)$. Hence,

$$\delta\|\xi\|^2 \leqslant \beta + 2M\|\xi\| + 2N\int_0^t \|\xi(u)\|\,du,$$

so that
$$(\|\xi\| - M\delta^{-1})^2 \leqslant \beta\delta^{-1} + M^2\delta^{-2} + 2N\delta^{-1}\int_0^t \|\xi\|\,du.$$

Set
$$\rho(t) \equiv \int_0^t \|\xi\|\,du, \quad A_1 \equiv \beta\delta^{-1} + M^2\delta^{-2}, \quad B_1 \equiv 2N\delta^{-1}.$$

Taking the square root of both sides of the above inequality, we obtain

$$\|\xi(t)\| = \dot{\rho}(t) \leqslant (A_1+B_1\rho)^{\frac{1}{2}} + M\delta^{-1}, \tag{4.6}$$

so that $$\frac{d}{dt}\{2B_1^{-1}(A_1+B_1\rho)^{\frac{1}{2}}\} = \dot{\rho}(A_1+B_1\rho)^{-\frac{1}{2}} \leqslant 1+M\delta^{-1}A_1^{-\frac{1}{2}}.$$

Integrating from 0 to t yields

$$(A_1+B_1\rho)^{\frac{1}{2}} \leqslant \tfrac{1}{2}B_1(1+M\delta^{-1}A_1^{-\frac{1}{2}})t + A_1^{\frac{1}{2}},$$

and (4.2) follows at once from (4.6). From (4.2) and (4.5) we find

$$(\dot{\xi}, P\dot{\xi}) \leqslant \beta + 2M(At+B) + 2N\int_0^t (Au+B)\,du,$$

which implies (4.3).

(ii) Let $\epsilon > 0$, and set $\xi(t) = e^{\epsilon t}\zeta(t)$. Then $\zeta(t)$ satisfies

$$P\ddot{\zeta} + K_\epsilon\dot{\zeta} + H_\epsilon\zeta + \tilde{F}_\zeta = f(t) \equiv re^{-\epsilon t} \quad (t \geqslant 0), \tag{4.7}$$

where $K_\epsilon \equiv 2\epsilon P + K \geqslant 0$, $H_\epsilon \equiv \epsilon^2 P + \epsilon K + H \geqslant 0$,

$$\inf_D \frac{(\zeta, H_\epsilon\zeta)}{(\zeta,\zeta)} > 0,$$

$\tilde{F}_\zeta \equiv F_\xi e^{-\epsilon t}$, and we have

$$(\tilde{F}_\zeta, \zeta) = (F_\xi, \xi)\,e^{-2\epsilon t} \equiv 0 \quad (t \geqslant 0), \tag{4.8}$$

$$(\tilde{F}_\zeta, \dot{\zeta}) = (F_\xi, \dot{\xi})\,e^{-2\epsilon t} - \epsilon(F_\xi, \xi)\,e^{-2\epsilon t} \equiv 0 \quad (t \geqslant 0). \tag{4.9}$$

Now $\|f\| = \|r\|e^{-\epsilon t} \leqslant M_\epsilon e^{\epsilon t}\,e^{-\epsilon t} = M_\epsilon$ and

$$\|\dot{f}\| = \|\dot{r} - \epsilon r\|e^{-\epsilon t} \leqslant (\|\dot{r}\| + \epsilon\|r\|)\,e^{-\epsilon t} \leqslant N_\epsilon + \epsilon M_\epsilon$$

so that by (i), $\|\zeta(t)\| \leqslant At+B$ for $t \geqslant 0$. Therefore, $\|\xi(t)\| = \|\zeta\|e^{\epsilon t} \leqslant (At+B)\,e^{\epsilon t}$ for $t \geqslant 0$, which implies that ξ is exponentially stable. Since r, $\dot{r}$, and ξ are exponentially stable, given any $\epsilon > 0$ there exist constants M_ϵ, N_ϵ, and $\tilde{M}_\epsilon$ such that $\|r\| \leqslant M_\epsilon e^{\epsilon t}$, $\|\dot{r}\| \leqslant N_\epsilon e^{\epsilon t}$, and $\|\xi\| \leqslant \tilde{M}_\epsilon e^{\epsilon t}$ for $t \geqslant 0$. It follows from (4.5) that

$$(\dot{\xi}, P\dot{\xi}) \leqslant \beta + 2M_\epsilon\tilde{M}_\epsilon e^{2\epsilon t} + 2N_\epsilon\tilde{M}_\epsilon\int_0^t e^{2\epsilon u}\,du$$
$$\leqslant [\beta + 2M_\epsilon\tilde{M}_\epsilon + \epsilon^{-1}N_\epsilon\tilde{M}_\epsilon]\,e^{2\epsilon t}$$

which proves the exponential stability of $(\dot{\xi}, P\dot{\xi})$.

(iii) This follows from (4.5) since $\|r\| = \|\dot{r}\| \equiv 0$.

Now consider the case where H is not non-negative on D. We introduce the following definitions:

$$\Delta \equiv \inf_D \frac{(\zeta, H\zeta)}{(\zeta,\zeta)} \quad (-\infty < \Delta < 0) \tag{4.10}$$

$$\tilde{D} \equiv \{\eta \,|\, \eta \text{ in } D,\ (\eta, H\eta) < 0\},$$

$$Q_\eta \equiv \frac{1}{2}\left\{\left[\frac{(\eta,K\eta)^2}{(\eta,P\eta)^2} - 4\frac{(\eta,H\eta)}{(\eta,P\eta)}\right]^{\frac{1}{2}} - \frac{(\eta,K\eta)}{(\eta,P\eta)}\right\} \quad \text{for} \quad \eta \text{ in } \tilde{D}, \tag{4.11}$$

(We assume that $P > 0$ on $\tilde{D}$, which holds for the problems of §§2 and 3.)

$$\Omega \equiv \sup_{\tilde{D}} Q_\eta,$$

$$K_\omega \equiv 2\omega P + K, \quad H_\omega \equiv \omega^2 P + \omega K + H.$$

Note that $\Omega > 0$, since $Q_\eta > 0$ for all η in $\tilde{D}$. Let $0 < \epsilon < \Omega$. Then there exists ϕ in $\tilde{D}$ such that $\Omega - \epsilon < Q_\phi \leqslant \Omega$. Now $(\phi, H_\omega \phi)$ is a strictly increasing function of ω on $[0, \infty)$ which vanishes for $\omega = Q_\phi$, so that $(\phi, H_{\Omega-\epsilon}\phi) < 0$. Let $\xi(t)$ be a solution of (4.1), with $r \equiv 0$ satisfying the initial data $\xi_0 = \phi$ and $\dot{\xi}_0 = (\Omega - \epsilon)\phi + \psi$, where ψ is some element of the nullspace of P (i.e. $P\psi = 0$). Then $\zeta(t) \equiv \xi(t)\, e^{-(\Omega-\epsilon)t}$ satisfies the equation,

$$P\ddot{\zeta} + K_{\Omega-\epsilon}\dot{\zeta} + H_{\Omega-\epsilon}\zeta + F_\zeta = 0, \tag{4.12}$$

where $F_\zeta \equiv F_\xi e^{-(\Omega-\epsilon)t}$, so that

$$\frac{d}{dt}\{(\dot{\zeta}, P\dot{\zeta}) + (\zeta, H_{\Omega-\epsilon}\zeta)\} = -2(\dot{\zeta}, K_{\Omega-\epsilon}\dot{\zeta}) \leqslant 0. \tag{4.13}$$

Therefore

$$\Delta\|\zeta\|^2 \leqslant (\dot{\zeta}, P\dot{\zeta}) + (\zeta, H_{\Omega-\epsilon}\zeta) \leqslant (\dot{\zeta}_0, P\dot{\zeta}_0) + (\zeta_0, H_{\Omega-\epsilon}\zeta_0) = (\phi, H_{\Omega-\epsilon}\phi),$$

which implies $\|\zeta\| \geqslant [(\phi, H_{\Omega-\epsilon}\phi)/\Delta]^{\frac{1}{2}}$. Hence

$$\|\xi(t)\| = \|\zeta(t)\| e^{(\Omega-\epsilon)t} \geqslant [(\phi, H_{\Omega-\epsilon}\phi)/\Delta]^{\frac{1}{2}} e^{(\Omega-\epsilon)t},$$

and this holds for *all* sufficiently small positive ϵ, i.e. the exponential growth rate Ω can be approached arbitrarily closely by solutions of the homogeneous equation. The general solution of (4.1) is the sum of a particular integral plus a solution of the homogeneous equation; it follows that the inhomogeneous equation (4.1) admits solutions with growth rates $\geqslant \Omega - \epsilon$ for every $\epsilon > 0$. (We say that $\xi(t)$ has a growth rate $\geqslant \omega$ if there exists a sequence $\{t_n\}_{n=1}^{\infty}$ with $t_n \to \infty$ as $n \to \infty$ such that $\|\xi(t_n)\| \geqslant M e^{\omega t_n}$ for some constant $M > 0$ and all n. If there is a constant N such that $\|\xi(t)\| \leqslant N e^{\omega t}$ for all $t \geqslant 0$, then $\xi(t)$ is said to have a growth rate $\leqslant \omega$.)

We now show that if the functions $r(t)\, e^{-\Omega t}$ and $\dot{r}(t)\, e^{-\Omega t}$ are both exponentially stable and if

$$\Delta \equiv \inf_D \frac{(\zeta, H_{\Omega+\epsilon}\zeta)}{(\zeta, \zeta)} > 0 \quad \text{for all} \quad \epsilon > 0,$$

then every solution of (4.1) has a growth rate $\leqslant \Omega + \epsilon$ for all $\epsilon > 0$, i.e. the growth rate Ω is never exceeded and is therefore the maximal growth rate of the system. (The preceding paragraph shows that solutions with growth rates arbitrarily close to Ω always exist, but no implication is made that the growth rate Ω is actually attained; indeed, in general it is *not* attained.) Let $\xi(t)$ be any solution of (4.1), let $\epsilon > 0$, and set $\zeta(t) \equiv \xi(t)\, e^{-(\Omega+\epsilon)t}$. Then $\zeta(t)$ satisfies the equation,

$$P\ddot{\zeta} + K_{\Omega+\epsilon}\dot{\zeta} + H_{\Omega+\epsilon}\zeta + F_\zeta = f(t) \equiv r e^{-(\Omega+\epsilon)t},$$

where $F_\zeta \equiv F_\xi e^{-(\Omega+\epsilon)t}$. Since $\|f\|$ and $\|\dot{f}\|$ are both bounded, it follows immediately from part (i) of theorem 4.1 that a constant M exists such that $\|\zeta(t)\| \leqslant M e^{\epsilon t}$ for all $t \geqslant 0$, and so

$$\|\xi(t)\| = \|\zeta(t)\| e^{(\Omega+\epsilon)t} \leqslant M e^{(\Omega+2\epsilon)t},$$

which was to be proved. We summarize our results in the following theorem:

Theorem 4.2. Let

$$\inf_D \frac{(\zeta, H\zeta)}{(\zeta, \zeta)} < 0.$$

(i) Then for each $\epsilon > 0$, there is a solution of (4.1) with a growth rate $\geqslant \Omega - \epsilon$.
(ii) Suppose, furthermore, that $re^{-\Omega t}$ and $\dot{r}e^{-\Omega t}$ are exponentially stable, and let

$$\inf_D \frac{(\zeta, H_{\Omega+\epsilon}\zeta)}{(\zeta,\zeta)} > 0 \quad \text{for all} \quad \epsilon > 0.$$

Then Ω is the maximal growth rate of the system. (A sufficient condition that

$$\inf_D \frac{(\zeta, H_{\Omega+\epsilon}\zeta)}{(\zeta,\zeta)} > 0 \quad \text{for} \quad \epsilon > 0$$

is that

$$\inf_D \frac{(\zeta, [\epsilon P + K]\,\zeta)}{(\zeta,\zeta)} > 0 \quad \text{for} \quad \epsilon > 0,$$

since $H_\Omega \geqslant 0$ on D and $H_{\Omega+\epsilon} = \epsilon[\epsilon P + K] + 2\Omega\epsilon P + H_\Omega \geqslant \epsilon[\epsilon P + K]$ on D for $\epsilon > 0$.)

5. Conclusions

(i) *The viscous fluid*

For this problem, P, K, and H are respectively equal to ρ_0, L_0, and G_0 (cf. (2.9) and (4.1)). We shall suppose that $\delta \equiv \inf_U \rho_0(\mathbf{x}) > 0$. Then if $\nabla\rho_0 . \nabla\Phi_0 \leqslant 0$ on U the exponential stability of $\boldsymbol{\xi}$ and $\mathbf{V}_1\,(= \dot{\boldsymbol{\xi}})$ follows immediately from theorem 4.1 (i), since $G_0 \geqslant 0$ by (2.14) and $\inf_D (\zeta, H_\alpha \zeta)/(\zeta,\zeta) \geqslant \alpha^2\delta > 0$ for $\alpha > 0$. A stronger result holds for the class of perturbations with initial data satisfying

$$\rho_1(\mathbf{x}, 0) = -\nabla\rho_0 . \boldsymbol{\xi}(\mathbf{x}, 0);$$

here we have $r(t) \equiv 0$, and it follows from theorem 4.1 (iii) that $\|\mathbf{V}_1(\mathbf{x}, t)\|$ is bounded. On the other hand, if $\nabla\rho_0 . \nabla\Phi_0 > 0$ at some point $\mathbf{x}$ in U, we may construct a $\boldsymbol{\xi}$ in D which makes the right-hand side of (2.14) negative, and it follows from theorem 4.2 that the system is exponentially unstable with maximal growth rate Ω given by

$$\Omega = \tfrac{1}{2} \sup_{\tilde{D}} \left\{ \left[\frac{(\boldsymbol{\xi}, L_0\boldsymbol{\xi})^2}{(\boldsymbol{\xi}, \rho_0\boldsymbol{\xi})^2} - 4\frac{(\boldsymbol{\xi}, G_0\boldsymbol{\xi})}{(\boldsymbol{\xi}, \rho_0\boldsymbol{\xi})} \right]^{\frac{1}{2}} - \frac{(\boldsymbol{\xi}, L_0\boldsymbol{\xi})}{(\boldsymbol{\xi}, \rho_0\boldsymbol{\xi})} \right\}. \tag{5.1}$$

Thus one-half the expression in curly brackets, evaluated for any $\boldsymbol{\xi}$ in $\tilde{D}$, furnishes a lower bound for Ω. A simple upper bound on Ω can be obtained by noting that if $\boldsymbol{\xi}$ is in $\tilde{D}$, the expression in curly brackets does not exceed

$$\left[\Delta^2 + 4\sup_{\tilde{D}}\left(-\frac{(\boldsymbol{\xi}, G_0\boldsymbol{\xi})}{(\boldsymbol{\xi}, \rho_0\boldsymbol{\xi})}\right)\right]^{\frac{1}{2}} - \Delta, \quad \text{where} \quad \Delta \equiv \inf_{\tilde{D}} \frac{(\boldsymbol{\xi}, L_0\boldsymbol{\xi})}{(\boldsymbol{\xi}, \rho_0\boldsymbol{\xi})},$$

and since

$$-\frac{(\boldsymbol{\xi}, G_0\boldsymbol{\xi})}{(\boldsymbol{\xi}, \rho_0\boldsymbol{\xi})} = \frac{\int_U \nabla\rho_0 . \nabla\Phi_0 |\xi_\parallel|^2\, d^3x}{\int_U \rho_0 |\boldsymbol{\xi}|^2\, d^3x} \leqslant \frac{\int_U \nabla\rho_0 . \nabla\Phi_0 |\xi_\parallel|^2\, d^3x}{\int_U \rho_0 |\xi_\parallel|^2\, d^3x}$$

$$\leqslant \sup_U \frac{\nabla\rho_0 . \nabla\Phi_0}{\rho_0}, \tag{5.2}$$

where $\xi_\parallel = (\boldsymbol{\xi}.\nabla\Phi_0)/|\nabla\Phi_0|$, we have

$$\Omega \leqslant \left[\frac{\Delta^2}{4}+\sup_U \frac{\nabla\rho_0.\nabla\Phi_0}{\rho_0}\right]^{\frac{1}{2}} - \frac{\Delta}{2} \leqslant \left[\sup_U \frac{\nabla\rho_0.\nabla\Phi_0}{\rho_0}\right]^{\frac{1}{2}}. \tag{5.3}$$

For an exponentially unstable inviscid ($L_0 \equiv 0$) fluid, Ω is actually equal to the extreme right-hand side of (5.3).† This follows from the fact that for any $\epsilon > 0$ and any $\mathbf{x}_0$ in the interior of U, we can construct a $\boldsymbol{\xi}(\mathbf{x})$ in D which vanishes outside the sphere $|\mathbf{x}-\mathbf{x}_0| < \epsilon$ and satisfies

$$\int_U |\boldsymbol{\xi}_\perp|^2\, d^3x \leqslant \epsilon \int_U |\xi_\parallel|^2\, d^3x > 0,$$

so that it is possible to choose $\boldsymbol{\xi}(\mathbf{x})$ in D so as to make $-(\boldsymbol{\xi}, G_0\boldsymbol{\xi})/(\boldsymbol{\xi}, \rho_0\boldsymbol{\xi})$ as close as we please to $\sup_U (\nabla\rho_0.\nabla\Phi_0/\rho_0)$. Thus unless

$$\frac{\nabla\rho_0.\nabla\Phi_0}{\rho_0} \equiv \sup_U \frac{\nabla\rho_0.\nabla\Phi_0}{\rho_0}$$

on some open sphere in U, Ω cannot be an eigenvalue in the inviscid case, i.e. a smooth solution of the form $\boldsymbol{\xi}(\mathbf{x})\, e^{\Omega t}$ to the homogeneous equation (2.9) with $L_0 = 0$ cannot exist.

Finally, we observe that if we write $\nu_0(\mathbf{x}) = \hat{\nu} g(\mathbf{x})$, where $\hat{\nu} \equiv \sup_U \nu_0(\mathbf{x})$, then

$$(\boldsymbol{\xi}, L_0\boldsymbol{\xi}) = \frac{\hat{\nu}}{2}\int_U g(\mathbf{x}) \left|\frac{\partial\xi_i}{\partial x_j}+\frac{\partial\xi_j}{\partial x_i}\right|^2 d^3x, \tag{5.4}$$

and it follows from (5.1) and the preceding paragraph that $\Omega(\hat{\nu})$ is a non-increasing (strictly decreasing if $\inf_U \nu_0(\mathbf{x}) > 0$) continuous function of $\hat{\nu}$ for $\hat{\nu} \geqslant 0$.

(ii) *The viscous resistive MHD fluid*

In this case stability is determined by the quadratic form (cf. (3.21) and (2.14))

$$(\bar{\xi}, H\bar{\xi}) = -\mu_0 \int_U \nabla\rho_0.\nabla\Phi_0 |\xi_\parallel|^2\, d^3x + \int_U |L_1^+\boldsymbol{\xi} - L_2\mathbf{R}|^2\, d^3x. \tag{5.5}$$

If $\nabla\rho_0.\nabla\Phi_0 \leqslant 0$ on U, then $H \geqslant 0$ on D and (2.5), (3.9), (3.13), (3.21), and (4.5) give

$$\mu_0\, \delta \int_U |\mathbf{V}_1(\mathbf{x},t)|^2\, d^3x + \int_U |\mathbf{B}_1(\mathbf{x},t)-\mathbf{C}|^2\, d^3x \leqslant \beta + 2\|\bar{r}\|\,\|\bar{\xi}\|, \tag{5.6}$$

where $\delta \equiv \inf_U \rho_0$ and $\mathbf{C}(\mathbf{x}) \equiv \mathbf{B}_1(\mathbf{x},0) + L_2\mathbf{R}(\mathbf{x},0) - L_1^+\boldsymbol{\xi}(\mathbf{x},0)$. It can be shown that (5.6) implies that if $\delta > 0$,

$$\int_U |\mathbf{V}_1(\mathbf{x},t)|^2\, d^3x \quad \text{and} \quad \int_U |\mathbf{B}_1(\mathbf{x},t)|^2\, d^3x$$

† For $\nu_0 \equiv 0$ the set D must be enlarged to require only that $\mathbf{f}.\mathbf{n} = 0$ on all parts of S not at infinity ($\mathbf{n}$ is the unit normal to S) rather than $\mathbf{f} = 0$ thereon.

are both bounded by polynomials of the fourth degree in t, so that the system is exponentially stable. Indeed, if we restrict our attention to perturbations whose initial data $\rho_1(\mathbf{x}, 0)$ and $\mathbf{B}_1(\mathbf{x}, 0)$ are given by

$$\left.\begin{aligned}\rho_1(\mathbf{x},0) &= -\nabla\rho_0.\boldsymbol{\xi}(\mathbf{x},0),\\ \mathbf{B}_1(\mathbf{x},0) &= L_1^+\boldsymbol{\xi}(\mathbf{x},0)-L_2\mathbf{R}(\mathbf{x},0),\end{aligned}\right\} \tag{5.7}$$

for some

$$\begin{pmatrix}\boldsymbol{\xi}(\mathbf{x},0)\\ \mathbf{R}(\mathbf{x},0)\end{pmatrix} \text{ in } D, \quad \text{then} \quad \mathbf{C}\equiv 0 \quad \text{and} \quad \bar{r}\equiv 0,$$

and it follows immediately from (5.6) that both

$$\int_U |\mathbf{V}_1(\mathbf{x},t)|^2\, d^3x \quad \text{and} \quad \int_U |\mathbf{B}_1(\mathbf{x},t)|^2\, d^3x$$

are bounded uniformly in time (provided $\delta > 0$).

On the other hand, if $\nabla\rho_0.\nabla\Phi_0 > 0$ at some interior point $\mathbf{x}_0$ of U and if η_0 is strictly positive in U, then it is possible (with reasonable assumptions on η_0, $\mathbf{B}_0$, and U) to construct a

$$\bar{\xi} = \begin{pmatrix}\boldsymbol{\xi}(\mathbf{x})\\ \mathbf{R}(\mathbf{x})\end{pmatrix}$$

in D for which $(\bar{\xi}, H\bar{\xi}) < 0$. Indeed, in the particular circumstance that a sphere $S_\epsilon(\mathbf{x}_0)$ of radius ϵ with centre at $\mathbf{x}_0$ exists within which $\mathbf{B}_0 = B_0\mathbf{e}_z =$ constant and η_0 is independent of z (without loss of generality we may also assume that $\nabla\rho_0.\nabla\Phi_0 > 0$ therein) we may construct such a $\bar{\xi}$ by simply setting

$$\mathbf{R} = \mu_0 B_0(\partial\mathbf{Q}/\partial z)$$

and $\boldsymbol{\xi} \equiv \nabla\times(\eta_0\nabla\times\mathbf{Q})$, where $\mathbf{Q} = \nabla\times\mathbf{W}$ and $\mathbf{W}$ is infinitely often differentiable and vanishes identically outside $S_\epsilon(\mathbf{x}_0)$. Then $\boldsymbol{\xi}$ and $\mathbf{R}$ vanish outside $S_\epsilon(\mathbf{x}_0)$, $\nabla.\boldsymbol{\xi} = \nabla.\mathbf{R} = 0$,

$$L_1^+\boldsymbol{\xi} = B_0\frac{\partial\boldsymbol{\xi}}{\partial z} = \nabla\times\left(\eta_0\nabla\times\left[B_0\frac{\partial\mathbf{Q}}{\partial z}\right]\right) = \mu_0^{-1}\nabla\times(\eta_0\nabla\times\mathbf{R}) = L_2\mathbf{R},$$

and thus if $\mathbf{W}$ is chosen so that $\xi_\parallel = [\nabla\times(\eta_0\nabla\times\mathbf{Q})].\nabla\Phi_0/|\nabla\Phi_0| \neq 0$ at $\mathbf{x}_0$, (5.5) implies that $(\bar{\xi}, H\bar{\xi}) < 0$. It follows from theorem 4.2 (i) that the system is exponentially unstable and that growth rates arbitrarily close to Ω can be achieved, where Ω is given by

$$\Omega = \tfrac{1}{2}\sup_{\bar{D}}\left\{\left[\frac{(\bar{\xi},K\bar{\xi})^2}{(\bar{\xi},\Pi\bar{\xi})^2} - 4\frac{(\bar{\xi},H\bar{\xi})}{(\bar{\xi},\Pi\bar{\xi})}\right]^{\frac{1}{2}} - \frac{(\bar{\xi},K\bar{\xi})}{(\bar{\xi},\Pi\bar{\xi})}\right\}. \tag{5.8}$$

Thus the necessary and sufficient condition for the exponential stability of a resistive ($\eta_0 > 0$ in U) fluid is that $\nabla\rho_0.\nabla\Phi_0$ be everywhere non-positive in U. This result differs strikingly with the situation for a perfectly conducting fluid; there it is well known, at least for simple geometries, that gravitational instabilities can be stabilized by sufficiently strong magnetic fields $\mathbf{B}_0$. Theorem 4.1 permits the following generalization to arbitrary geometries.

Theorem. Let the region U be filled with a viscous, incompressible, perfectly conducting fluid satisfying (3.1)–(3.5) (with $\eta \equiv 0$). Let the equilibrium quantities ρ_0, Φ_0, and P_0 satisfy (2.4), $\mathbf{V}_0 \equiv 0$, $\inf_U \rho_0 > 0$, $\nu_0 > 0$ on U and its boundary S, $\mathbf{B}_0 = B_0 \mathbf{e}_z =$ constant, and $B_0 > 0$. We suppose that S is a perfectly conducting rigid surface, and require that the perturbed quantities $\mathbf{V}_1(\mathbf{x}, t)$ and $\mathbf{B}_1(\mathbf{x}, t)$ satisfy the boundary conditions $\mathbf{V}_1 = 0$ on S, $\mathbf{B}_1 = 0$ on all parts of S at infinity, and $\mathbf{B}_1 . \mathbf{n} = 0$ on the remainder of S. Finally, let $\Lambda < \infty$ be the maximum length of U in the direction $\mathbf{e}_z$. Then, if

$$\frac{\mathbf{B}_0^2 \pi^2}{\mu_0 \Lambda^2} > \sup_U \boldsymbol{\nabla}\rho_0 . \boldsymbol{\nabla}\Phi_0,$$

the equilibrium is stable, i.e.

$$\int_U |\mathbf{V}_1(\mathbf{x}, t)|^2 \, d^3x \quad \text{and} \quad \int_U |\mathbf{B}_1(\mathbf{x}, t)|^2 \, d^3x$$

are both uniformly bounded for $t \geqslant 0$.

Proof. The perturbed quantities satisfy the equations

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} + L_0 \frac{\partial \boldsymbol{\xi}}{\partial t} + H\boldsymbol{\xi} + \boldsymbol{\nabla} P_1 = \mathbf{r}, \tag{5.9}$$

and
$$\mathbf{B}_1(\mathbf{x}, t) = L_1^+ \boldsymbol{\xi} + \mathbf{B}_1(\mathbf{x}, 0) - L_1^+ \boldsymbol{\xi}(\mathbf{x}, 0), \tag{5.10}$$

where $\mathbf{r} \equiv -[\rho_1(\mathbf{x}, 0) + \boldsymbol{\nabla}\rho_0 . \boldsymbol{\xi}(\mathbf{x}, 0)] \boldsymbol{\nabla}\Phi_0 + \mu_0^{-1} L_1 [L_1^+ \boldsymbol{\xi}(\mathbf{x}, 0) - \mathbf{B}_1(\mathbf{x}, 0)],$

$$H \equiv G_0 + L_1 L_1^+ \mu_0^{-1},$$

and the remaining quantities are as defined previously. The quantities $\boldsymbol{\xi}$ and $\partial\boldsymbol{\xi}/\partial t$ are required, for each fixed $t \geqslant 0$, to be in D, the set of all twice continuously differentiable functions $\mathbf{f}(\mathbf{x})$ on U satisfying $\nabla . \mathbf{f} = 0$ and the boundary conditions imposed on $\mathbf{V}_1$ in the hypothesis. The operator H is self-adjoint on D, and we show that $\inf_D (\boldsymbol{\xi}, H\boldsymbol{\xi})/(\boldsymbol{\xi}, \boldsymbol{\xi}) > 0$. Now

$$\begin{aligned}(\boldsymbol{\xi}, H\boldsymbol{\xi}) &= (\boldsymbol{\xi}, G_0 \boldsymbol{\xi}) + \mu_0^{-1} \|L_1^+ \boldsymbol{\xi}\|^2 \\ &= -\int_U \boldsymbol{\nabla}\rho_0 . \boldsymbol{\nabla}\Phi_0 |\xi_\parallel|^2 \, d^3x + \mu_0^{-1} B_0^2 \int_U \left|\frac{\partial \boldsymbol{\xi}}{\partial z}\right|^2 d^3x,\end{aligned}$$

and
$$\int_U \left|\frac{\partial \boldsymbol{\xi}}{\partial z}\right|^2 d^3x = \sum_{i=1}^3 \int_U \left|\frac{\partial \xi_i}{\partial z}\right|^2 d^3x \geqslant \frac{\pi^2}{\Lambda^2} \|\boldsymbol{\xi}\|^2,$$

since
$$\begin{aligned}\int_U \left|\frac{\partial \xi_i}{\partial z}\right|^2 d^3x &= \iint \left[\int \left|\frac{\partial \xi_i}{\partial z}\right|^2 dz\right] dx\, dy \\ &\geqslant \iint \left[\frac{\pi^2}{\Lambda^2} \int |\xi_i|^2 \, dz\right] dx\, dy \\ &= \frac{\pi^2}{\Lambda^2} \int_U |\xi_i|^2 \, d^3x.\end{aligned}$$

Therefore
$$(\boldsymbol{\xi}, H\boldsymbol{\xi}) \geqslant \frac{B_0^2 \pi^2}{\mu_0 \Lambda^2} \|\boldsymbol{\xi}\|^2 - \int_U \boldsymbol{\nabla}\rho_0 . \boldsymbol{\nabla}\Phi_0 |\xi_\parallel|^2 \, d^3x.$$

If $\sup_U \boldsymbol{\nabla}\rho_0 . \boldsymbol{\nabla}\Phi_0 \leqslant 0$, the result is immediate; if $\sup_U \boldsymbol{\nabla}\rho_0 . \boldsymbol{\nabla}\Phi_0 > 0$, then we have

$$(\boldsymbol{\xi}, H\boldsymbol{\xi}) \geqslant \left(\frac{B_0^2 \pi^2}{\mu_0 \Lambda^2} - \sup_U \boldsymbol{\nabla}\rho_0 . \boldsymbol{\nabla}\Phi_0\right) \|\xi\|^2,$$

and again we conclude that $\inf_D (\boldsymbol{\xi}, H\boldsymbol{\xi})/(\boldsymbol{\xi}, \boldsymbol{\xi}) > 0$. It follows at once from theorem 4.1 (i) that $\|\boldsymbol{\xi}\|$ and

$$\left\| \frac{\partial \boldsymbol{\xi}}{\partial t} \right\| = \|\mathbf{V}_1\|$$

are bounded for all $t \geqslant 0$; (4.5) then implies the boundedness of $(\boldsymbol{\xi}, H\boldsymbol{\xi})$ for $t \geqslant 0$, from which we infer that $\|L_1^+ \boldsymbol{\xi}\|$ and therefore $\|\mathbf{B}_1\|$ are also bounded in t for all $t \geqslant 0$.

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